Symmetric heaping in grains: A phenomenological model

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Heap formation of granular materials in a vertical vibrating bed is studied by a simple model using the profile of the heap as the dynamic variable. Vibration increases the local height, but is counterbalanced by the nonlinear coupling, which tends to suppress the growth of the height. The steady state heap can be solved in closed form in terms of Jacobian elliptic functions. Phenomena such as heap formation and downward and upward heaps can be reproduced. Our results agree with the experimentally observed change of downward to upward steady heaps as the vibration strength is increased. Predictions from the model compare favorably with experimental results on heap profiles and heaping angles.

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I. INTRODUCTION

Granular materials has been the subject of great scientific and engineering interest for many years [1]. These granular materials have the unique mechanical properties that they cannot be easily classified as either solid or liquid since they can sustain shear like a solid as well as flowing like a liquid when driven. However, as distinct from liquids, granular heaps (sandpiles) are stable as long as the top surface is at a slope less than the angle of repose. When the slope is increased slightly above the maximum angle of stability, grains begin to flow and an avalanche of particles occurs. Because of this unique characteristic, granular materials exhibit many unusual behavior in a vertical vibrating bed such as size segregation [2–4], density waves [5], convective transport [6], and spectacular pattern formations [7,8].

One of the most remarkable phenomena in granular materials is the heap formation of grains in a vertically shaken container [9–11]. The relevant dimensionless parameter for this problem is the reduced acceleration amplitude $\Gamma = A\omega^2/g$ of the bed, where g, A, and ω are the gravitational acceleration, amplitude, and angular frequency of the vibrating bed, respectively. An originally flat layer of sand will become unstable and turn into a heap with a well-defined structure if Γ is greater than some threshold value Γ_c. Heap formation is most spectacular when the number of grains is relatively small. In such a case, an asysmetric heap will be formed with most of the grains gathered in only one part of the container breaking the symmetry of the original system. The grains in the heap are not simply moving up and down vertically, but a convection roll with grains moving up along the wall and flows down the slope of the heap is also produced. On the other hand, when the number of grains is large enough, it has been shown experimentally [12] that two types of symmetric heaps can be formed, namely, the upward (valley) and downward (mountain) modes. The main difference between these two modes is that the convection current of the granular particles next to the wall move up in the upward mode, and vice versa. It is observed that as Γ increases, the downward mode is formed first and followed by the upward mode. So far there is no theoretical explanation for the upward heap, nor a fundamental understanding of the role played by boundaries (walls) in this case.

Friction between the grains and with the wall, driving mechanism, and boundary effects [6,13] all play important roles in the heap formation. There are many investigations both in theory [2,14–16] and experiment [10,12,17] as well as numerical simulation [18,19] to investigate the underlying mechanism of heap formation in a vertically vibrating granular layer. One of the main issues is to understand the nature of the instability at the threshold of heap formation where the vibration strength is still relatively low, and to relate the observed phenomena to some vital dynamic granular properties yet to be identified. One plausible candidate is the density fluctuations induced by vibration in the presence of interstitial gases. It is found that interstitial gas is vital for the heap formation. Pak *et al.* established in their experiments [17] that heaping and convection current are strongly suppressed when there is no interstitial gas. Therefore, density fluctuations induced by vibration seem to be essential for heap formation, and any successful heaping model should also include this effect. Since the overall decrease in the density and density fluctuations of grains under vertical vibrations can be viewed as the creation of voids in the granular materials, there have been some models to include void dynamics [14,16], but they do not necessarily lead to the formation of heaps. Furthermore, these models of voids are quite complicated cellular automata models, and there is no analytical result at all. On the other hand, the sidewalls are also essential, since they confine the grains to flow in a finite region and the frictional properties of the wall can affect the convection mode.

In this paper, we take another theoretical approach in developing a phenomenological model for the formation of heaps based on the ideas of void models. The key point is the observation that the presence of void in a granular medium would modify and/or increase the effective height of the heaping profile. By using the height profile as the only dynamical variable, and taking into account the decrease in local density due to vibration and the nonlinear couplings for

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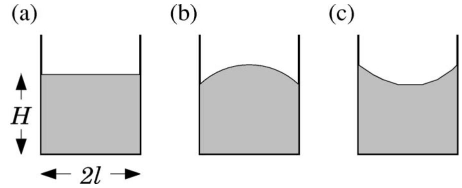

FIG. 1. Schematic pictures of a granular layer in a vibrating bed. (a) Initial flat layer of size $L\times H$. (b) Downward heap. (c) Upward heap.

energy dissipation, we introduce a simple mean field model which can reproduce many of the observed phenomena such as heap formation and downward and upward modes. Our model is aimed at predicting the structures and dynamics of formation of steady state heaps in term of a surface profile of the heap.

II. HEAP EQUATION

Our model is based on two important observations in vibrating bed experiments. The first is that energy is pumped into the medium by vibration, which causes density fluctuations and causes the layer to expand. Second, the grains roll down the slope by surface flow and cause the profile to flatten. We shall consider a quasi-two-dimensional vibrating bed for simplicity, as it can be easily extended to three dimensions. The simplest phenomenological model one can construct is by using the profile of the layer as a dynamical variable. Let $h(x,t)$ denotes the height of the sandpile at position x and time t, the equation of motion of $h(x,t)$ is proposed to be

$$\frac{\partial h}{\partial t}=D\frac{\partial^2 h}{\partial x^2}+\Omega h-\beta h^2, \tag{1}$$

where D is a diffusion constant which approximates the surface relaxation of fluctuations in h, and Ωh is the effect of increase in height due to vibrations per unit time. Equation (1) models the situation when the layer becomes unstable and tends to expand for sufficient vibration accelerations ($\Gamma>1$). It is obvious that the $-\beta h^2$ term is the decrease in height per unit time, which can be interpreted as the rate of dissipation of energy in the system. The two terms $\Omega h-\beta h^2$ can then be regarded as the first two terms of the Taylor expansion of some nonlinear interaction. The h^2 term signifies nonlinear couplings of different modes in the system. Presumably higher-order terms will enter for sufficiently strong nonlinear effects. However, for simplicity, we will stick to Eq. (1). The system of interest consists of N grains, each of size a, in a quasi-two-dimensional container of width $2l$. The midpoint of the bottom of the container is taken to be the origin (see Fig. 1). We shall assume that N is sufficiently large that the bottom of the container is always covered with grains. The major interest in this paper will be focused on the steady state profile of the granular layer. We shall assume that the initial profile is symmetric, $h(-x,0)=h(x,0)$, for convenience. Since the conditions of the left and right walls are assumed to be identical and will not change with time, the steady state profile must be symmetric. The steady state profile will be denoted by $h_s(x)$. Therefore, one of the boundary condition for h_s is

$$h_s(-x)=h_s(x). \tag{2}$$

The other boundary condition for this problem is based on the observation that the volume of the steady state heap is expanded somewhat so that the total volume of the layer is $Na^2[1+\alpha(\Omega,A)]$, where $\alpha(\Omega,A)$ is an expansion coefficient which is in general depending on the vibration frequency and amplitude. Since we are interested in the heaping phenomenon in which the vibrational acceleration is small, $\alpha\ll 1$. This has also been observed from experiments [20] and from our simulation model [21], that α is of the order of 10^{-2} to 10^{-3}. Hence we shall impose the boundary condition that the total volume of the layer is conserved:

$$\int_{-l}^{l} h_s(x)\,dx=Na^2. \tag{3}$$

This boundary condition can also be rationalized from the following point of view: imagine that after reaching the steady state, the profile of the layer is measured after the external vibration is turned off suddenly, the granular layer is then frozen, the interstitial gases then escape from the layer. In this situation, the above boundary condition would hold.

The dynamics for the simple linear case when $\beta=0$ can be solved exactly. In this case, Eq. (1) becomes the linear diffusion type equation

$$\partial_t h=D\partial_x^2 h+\Omega h. \tag{4}$$

Then with the symmetric and conserved volume boundary conditions similar to Eqs. (2) and (3), the solution for an initially flat profile $h(x,0)=H=Na^2/(2l)$ is solved to be

$$h(x,t)=\frac{Hkl}{\sin kl}\left(\cos kx+2kl\sin kl\sum_{m=1}^{\infty}\frac{(-1)^m\cos\frac{m\pi}{l}x}{(m\pi/l)^2-k^2}\times e^{-[(m\pi/l)^2-k^2]Dt}\right), \tag{5}$$

where $k\equiv\sqrt{\Omega/D}$. The dynamics for the non-linear $\beta\neq 0$ case can be solved numerically and the relation with other problems of the dynamics of interfaces will be presented elsewhere [22].

III. STRUCTURES OF STEADY HEAPS

In this paper, we shall focus on the structure of steady heaps produced from Eq. (1). With $\partial h/\partial t=0$, Eq. (1) becomes an ordinary differential equation with boundary conditions (2) and (3).

A. Linear model

For the linear case, the steady state profile obeys

$$\frac{d^2 h_s}{dx^2}+k^2 h_s=0. \tag{6}$$

$1/k$ represents the characteristic length scale of the heap. Some general structural properties can be obtained from boundary condition (3) before solving Eq. (6). Integrating Eq. (6) and using Eq. (3), one obtains

$$h_s'(l)=h_s'(-l)-Nk^2a^2. \tag{7}$$

Thus in the linear regime, it is impossible to have a upward heap (valley) since such a heap has $h_s'(l)>0$ and $h_s'(-l)<0$, which violates the above relation. Equation (6) is solved easily with boundary conditions (2) and (3), to give

$$h(x)=\frac{Nka^2}{2\sin kl}\cos kx. \tag{8}$$

One of the experimentally measurable quantities that characterizes the heap is the heaping angle

$$\Theta_H=\tan^{-1}(h'(-l)). \tag{9}$$

Our model in the linear regime gives

$$\tan\Theta_H=\frac{1}{2}Nk^2a^2. \tag{10}$$

Obviously, the result from the linear model would make sense only in the small k regime. In fact Eq. (8) implies that h is positive for all x only for $0\leqslant k<\pi/(2l)$. As the vibration becomes stronger, nonlinear effects must be important.

B. Nonlinear model

In the steady case, the heaping equation (1) becomes

$$\frac{d^2h_s}{dx^2}+k^2h_s-\beta' h_s^2=0 \tag{11}$$

where $\beta'\equiv\beta/D$. Equation (11), together with boundary conditions (2) and (3) can be solved in closed form in terms of generalized elliptic functions (see the Appendix for details) to give

$$h_s(x)=h_0+(h_0-u_+)\mathrm{tn}^2\{x\sqrt{\beta'(h_0-u_-)/6}|m)\}, \tag{12}$$

where $m\equiv(u_+-u_-)/(h_0-u_-)$, tn is the tangent Jacobian elliptic function [23], $h_0\equiv h_s(0)$, $2u_\pm\equiv(3k^2/2\beta')-h_0\pm\sqrt{[(3k^2/2\beta')-h_0][(3k^2/2\beta')+3h_0]}$. The value of h_0 is determined from Eq. (3), and can be solved from

$$2h_0l-\frac{12v}{\beta' l}[E(\mathrm{am}v|m)-\mathrm{dn}(v|m)\mathrm{tn}(v|m)]=Na^2, \tag{13}$$

where $v\equiv l\sqrt{\beta'(h_0-u_-)/6}$. The heaping angle is given by

$$\tan\Theta_H=(u_+-h_0)\sqrt{2\beta'(h_0-u_-)/3}\,\mathrm{dn}(v|m)\times\mathrm{tn}(v|m)/(\mathrm{cn}(v|m))^2 \tag{14}$$

Before we present the exact solution of $h_s(x)$, much insight can be obtained using a perturbative approach by considering the case of small nonlinearity. For small β', Eq. (11) can be solved perturbatively to give

$$h_s(x)=h_0\cos kx-\frac{\beta' h_0^2}{k^2}\left[\frac{\cos kx}{3}+\frac{\cos 2kx}{6}-\frac{1}{2}\right]+\frac{\beta'^2h_0^3}{k^4}\left[\frac{29\cos kx}{144}+\frac{\cos 2kx}{9}+\frac{\cos 3kx}{48}+\frac{5kx\sin kx}{12}-\frac{1}{3}\right]+O(\beta'^3), \tag{15}$$

and h_0 is given by

$$h_0=\frac{kNa^2}{2\sin(kl)}+\frac{N^2a^4\eta(kl)}{4\sin^3(kl)}\beta'+\frac{N^3a^6}{8k\sin^4(kl)}\times\left(\frac{2\eta^2(kl)}{\sin(kl)}-\frac{89\sin(kl)}{144}-\frac{\sin(2kl)}{18}-\frac{\sin(3kl)}{144}+\frac{5kl\cos(kl)}{12}+\frac{kl}{3}\right)\beta'^2+O(\beta'^3) \tag{16}$$

where $\eta(x)\equiv(\sin x/3)+[\sin(2x)/12]-x/2$. One can see directly the systematic emergence of smaller wavelength modes for higher orders in β'. In other words, a steady heap of finer spatial structure appears at stronger nonlinearity. The nonlinear interaction causes different modes to couple, and results in some steady spatial pattern.

One expects nonlinearity to become more important as the vibration becomes stronger, i.e., β' should increase with k. Furthermore, the only length scale that depends on the strength of vibration in the system is $1/k$, the length scale of other smaller structures are fractions of $1/k$, as suggested from the pertubative solution. Hence it is reasonable to assume the only frequency dependent length scale in the system is $1/k$. From dimensional analysis, β' must be of the form μk^3, where μ is a dimensionless parameter controlling the strength of the leading nonlinear effect. The steady state profiles are shown in Fig. 2(a) for various values of k. In addition, the pertubation results of $h(x)$ are also shown in Fig. 2(b) for comparison.

Starting with a flat layer of N grains in a system of width $2l$, the steady state behavior of the system is monitored for various values of vibration strength k. As shown in Fig. 2, the steady state heap changes from downward (mountain) modes to upward modes as k increases. A characteristic measure of heaping is the ratio $h(0)/H$, where H is the height of the originally flat grains. Hence $h(0)/H>1$ and <1 correspond to the downward and upward heaps, respectively. Figure 3 shows the variation of $h(0)/H$ as a function of k for a given value of μ, the change in the morphology of the steady heap from the downward to the upward modes is clear as k increases. The transition from downward to upward heap has also been observed in recent experiments [12] as Γ increases. Our recent cellular automata model based on the sandpile model plus empty site dynamics, also revealed a similar phenomenon as the vibration strength is increased.

C. Effect of absolute system size: scaled heaping profiles

To investigate the system size effects on the steady heaps, it is convenient to introduce some scaled variables,

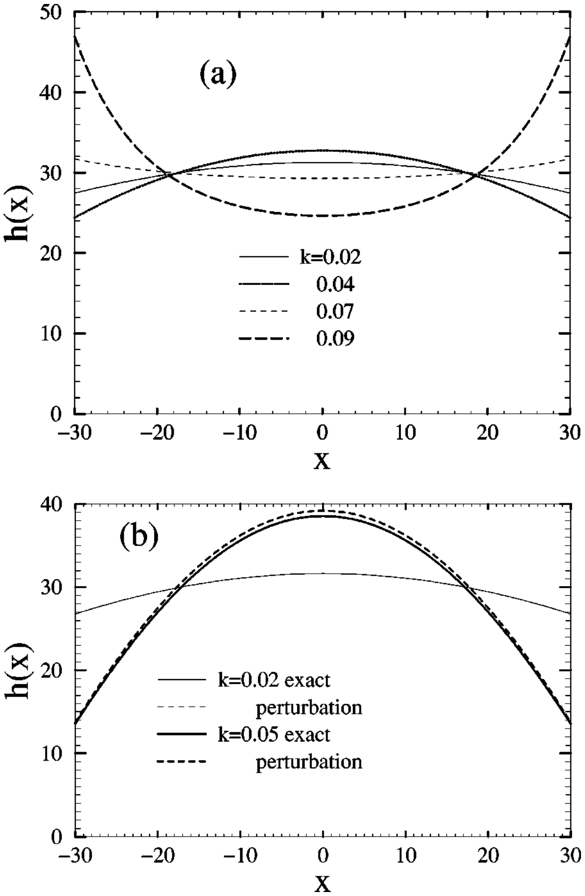

FIG. 2. Profile of steady heaps for a system with $N=1800$ and $L/a=60$ for various values of vibration strength k. (a) $h(x)$ calculated from Eq. (12) with $\mu=0.5$. (b) $h(x)$ calculated exactly from Eq. (12) and by pertubation using Eq. (15), with $\mu=0.2$. h and k are in units of a and $1/a$, respectively.

$$\tilde{x}\equiv x/l, \quad \tilde{h}\equiv h/H, \tag{17}$$

and define the aspect ratio χ of the original flat layer as $\chi\equiv H/(2l)$. Then the steady state heap equation becomes

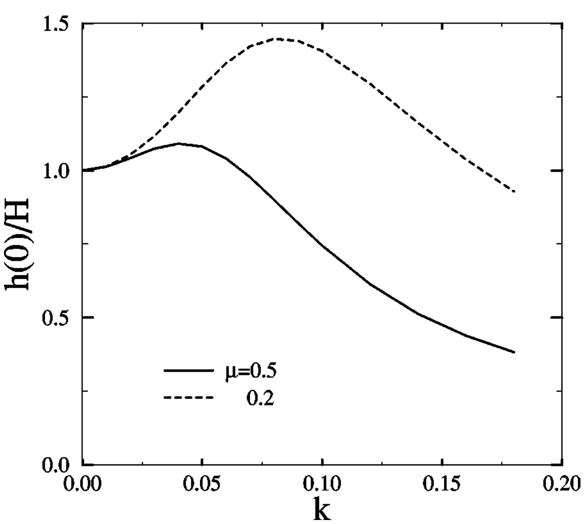

FIG. 3. $h(0)/H$ vs k with different values of μ. $N=1800$ and $L/a=60$. k is in unit of $1/a$.

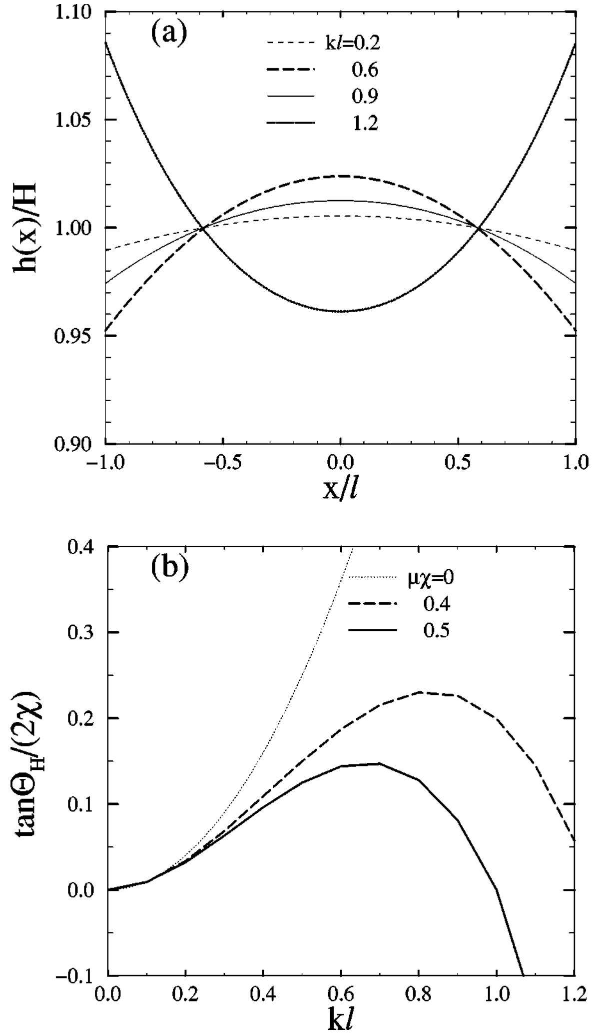

FIG. 4. (a) Scaled steady heap profiles $\tilde{h}(\tilde{x})$ with $\mu\chi=0.5$ for various values of scaled vibration strength kl. (b) Scaled heaping angle $\tan\Theta_H/(2\chi)$ vs kl for various values of $\mu\chi$.

$$\tilde{h}_s''(\tilde{x})+(kl)^2\tilde{h}_s(\tilde{x})-2\mu\chi(kl)^3\tilde{h}_s^2(\tilde{x})=0. \tag{18}$$

and the boundary conditions become $\int_0^1\tilde{h}_s(\tilde{x})d\tilde{x}=1$ and $\tilde{h}_s'(0)=0$. Thus the scaled profile $\tilde{h}_s(\tilde{x})$ depends only on the values of kl and $\mu\chi$. Figure 4(a) displays the scaled profiles for different values of kl with a fixed value of $\mu\chi=0.5$. It can be easily seen from Eq. (18) that a downward steady heap will change to an upward heap when the vibration strength is increased to the threshold value given by

$$k_c l=1/(2\mu\chi). \tag{19}$$

The degree of heaping can best be quantified by the heaping angle, and is given in terms of the scaled variable, as

$$\tan\Theta_H=2\chi\tilde{h}_s'(-1). \tag{20}$$

Figure 4(b) shows $\tilde{h}_s'(-1)$ as a function of kl for different values of $\mu\chi$, including the $\mu\chi=0$ (the linear model) case. A nonvanishing μ (i.e., nonlinear suppression of height) is essential to avoid the unphysical behavior of indefinite growing of the heaping angle at large vibrations. At first sight, it

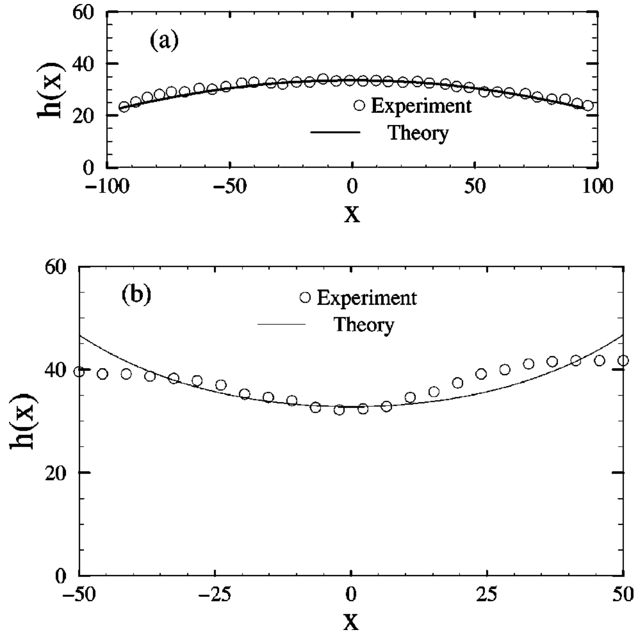

FIG. 5. Profile of heap from theory and experiments. h and x are in units of mm. (a) Downward heaping profile. Data are from the experiment in Ref. [20]; $\Gamma=1.5$ (glass beads of diameter $a=3$ mm, $H=30$ mm and $L=190$ mm). (b) Upward heaping profile. Data are from the experiment in Ref. [12]; $\Gamma=5.9$ (glass beads of diameters $a=0.61$ mm, $H=37$ mm and $L=100$ mm).

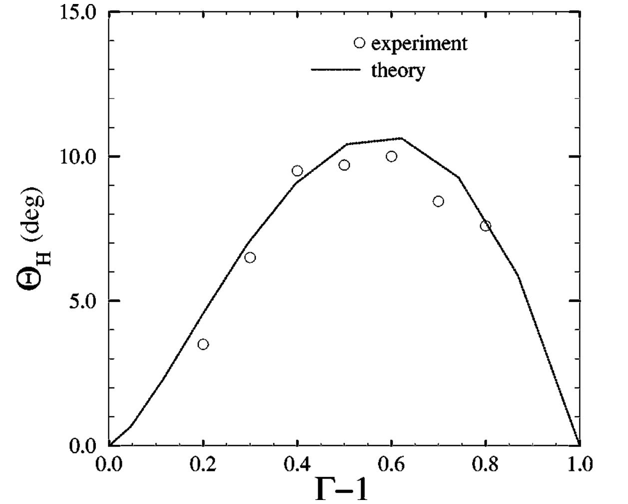

FIG. 6. Experimental data of heaping angles Θ_H of the steady profile vs $\Gamma-1$ for a system of glass beads with a diameter $a=1.28$ mm; $H/a\simeq 54.7$ and $L/a=86$. The curve is the fitting from our analytical prediction Eq. (14)] using $\mu=0.78$.

appears that a system with a larger absolute size is easier to excite (can form a heap of the same Θ_H with a smaller value of k) than a smaller system of the same aspect ratio. However such a naive analysis is complicated by the fact that μ is actually size dependent. Since μ represents the degree of suppression of the growth of the layer due to dissipation, without taking into account the constant area boundary condition, from a simple consideration one expects that, for small μ, the average height would be very large due to the small dissipation and height suppression, while the average height would be very small for very large μ. Thus it is reasonable to assume that μ is a decreasing function of H. This assumption will be verified in Sec. IV, when the values of μ are obtained from fitting with experimental data. Hence for two systems of different absolute sizes but with the same aspect ratio χ, a larger system would have a smaller μ, and could be excited to a steeper heap for the same degree of vibration strength k [see Fig. 4(b)]. Thus the conclusion that a larger system is easier to excite is still true, and this also agrees with experimental observations [12,20].

IV. COMPARISON WITH EXPERIMENTS

One of the remarkable predictions of such a simple model is that both downward (mountain) and upward (valley) modes of steady heaping can be obtained as k is varied. Such a change in the morphology of the steady heap profiles as the vibration strength is increased was also observed in experiment [12], and also in our recent simulation of a simple void model [21]. Using Eq. (11), one can also solve for steady state heaping profiles to compare with real experiments. Figure 5(a) shows the steady downward heap profile obtained from a quasi-two-dimensional heaping experiment [20] with $\Gamma=1.5$, together with the calculated profile from our equation. It can be seen that the theoretical prediction agrees well with the experimental profile. Also the steady upward heap profile in Ref. [12] is also fitted reasonably well with our calculated profile [Fig. 5(b)]. Larger deviations near the two walls are probably due to corrections from higher order nonlinear effects, i.e., at such large vibrations, the sytems may tend to produce more pairs of rolls and hence the slope near the walls are smaller than predicted by the theory.

The heaping angles of steady heap profiles were also measured as a function of Γ in recent vibrating bed experiments [20]. These measured angles are compared with our calculated values obtained from Eq. (14). In order to identify the experimental values of $\Gamma\equiv A\omega^2/g$ with the vibrational strength in our model, we consider the following scaling analysis. Supposing the bed vibrates vertically as $x=A\sin\omega t$, and that the granular layer is excited when $\Gamma>1$, and assuming that in the steady excited state there is maximum adsorption of energy in the layer (or a constant fraction of the maximum energy adsorbed), then one can easily calculate that the work done by the bed in one period is $\propto A^2\omega^3$. This latter would be the input power to the layer in the heap equation, and thus $\sim\Omega$. Thus one has $A^2\omega^3$ scales as (depends linearly on) Ω, since $k^2\propto\Omega$ and $k=0$ correspond to $\Gamma=1$ (no expansion). One finally arrives at the identification

$$\Gamma-1\propto k^{4/3}. \tag{21}$$

Figure 6 displays the measured data together with our theoretical results with the above identification. The theoretical curve is obtained by fitting the parameter μ and the proportional constant in Eq. (21). It should be noted that the peak value of the curve depends only on the fitted value of μ. To investigate further the effect of the absolute size of the layer on the degree of heaping, we further analyze the experimental data on heaping angle for systems of different initial bed heights H but with the same L. Figure 7 shows Θ_H as a function of $\Gamma-1$, together with the fitting curves of the the-

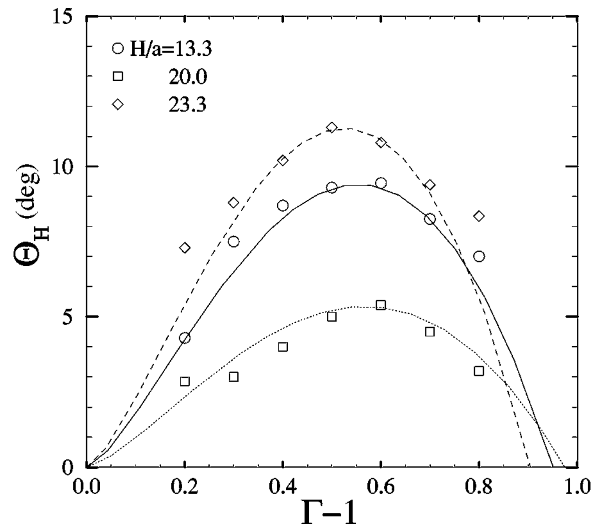

FIG. 7. Experimental data of heaping angles Θ_H of the steady profile vs $\Gamma-1$ for a system of glass beads with a diameter $a=3$ mm with fixed $L/a=36.7$ and different values of H/a. The curves are the fittings from our analytical prediction [Eq. (14)].

oretical results. It is clear from the data that a thick layer is easier to excite to a steeper heap. For layers with $H/a=13.3$, 20, and 23.3, the fitted values of μ are 1.48, 0.904, and 0.763, respectively. Thus we verify the assumption of Sec. III that μ is a decreasing function of H. Furthermore, for these layers, with Eq. (19) one can compute the theoretical threshold value of k_c above which the heap become upward (a valley); we obtain $k_c=0.935$, 1.014, and 1.031, respectively for the layers with $H/a=13.3$, 20, and 23.3. In other words, we deduce that the threshold vibrational strength for the morphology change increases with H (for systems with fixed L), which was also observed in another experiment [12].

V. CONCLUSION

In this paper, a simple continuum model is introduced for the formation of granular heaps in vertically vibrating beds. Despite the simplicity of the model, it successfully accounts for the structure of both downward and upward modes of heaps, and it also compares well with experimental data on heap profiles and heaping angles. Our model is simple enough that it allows for an analytical solution for the steady state profile. Such an analytical result is important, and can provide valuable insight into the system which is otherwise almost impossible since numerical simulations usually have to be employed for such complex nonlinear systems. It should be remarked that our continuum approach does not account for the descrete nature of the granular particles, and we anticipate that our results would describe experiments more accurately for finer grains.

It must be pointed out that our model works only for symmetric heaps, but does not describe the formation of the one sided heap discussed in Sec. I. One of the main differences in the formation of the symmetric and one sided heap is the number of grains in the system. In order to form a one sided heap, the number of grains must be small so that $h(x,t)$ can become zero quite easily even during relatively weak vibration. In the early stage of one sided heap formation, the process can probably still be described by our heaping equation until the small fluctuations are amplified by the nonlinear interaction to give $h(x,t)=0$. In a steady state one sided heap, there is an extended range of x such that $h=0$. Presumably, some symmetry breaking processes due to random fluctuations must set in, and are amplified by some unstable mechanism. In such a case, the size of the heap is not determined by the system size but rather the number of grains. Since the steady state profile close to the threshold of instability will be always symmetric and almost flat, our model would give an accurate prediction for $h(x,t)$ not very far from a flat profile, In this sense, one can also extend our equation in an *ad hoc* manner to include the case with $\Gamma<1$ by

$$\partial h/\partial t=D\partial^2 h/\partial^2 x+\Theta(\Omega-\Omega_c)f(h), \tag{22}$$

where $\Theta(\Omega-\Omega_c)$ is the Heaviside function which will not be zero only when $\Omega>\Omega_c$. Thus one can identify Ω_c as Γ_c and $f(h)=(\Omega-\Omega_c)h-\beta h^2+\cdots$. The success of our model for symmetric heaps is due to its ability to capture the two main effects in the system, namely, the input and the dissipation of energy. Energy is put into the system by the increase in height of the system, while dissipation is represented by the $-\beta h^2$ term that removes the potential energy. The heaping instability occurs when the energy injected into the system by vibration cannot be dissipated fast enough. Bulk flow will occur similarly to thermal convection. Our heaping equation describes how the flow interacts nonlinearly with the external vibration through the mean field description of $h(x,t)$. Steady states with heap formation can be reached when the system balances these two effects. It should be noted that the boundary also plays a crucial role by confining the grains in a finite region, and leads to convection. In our model, one can define an effective one-dimensional current [22] from Eq. (1), and the vanishing of this effective current at the hard walls leads to the boundary condition (3). It can also be seen from our model that vibration is treated as a mean field, with its effect averaged over one vibration cycle. Its mere effect is to produce input energy by increasing the height. In response to these increases, the profile dynamics interact to relax the fluctuations and also suppress the increase in height by dissipation. Therefore, our model is more or less a pure relaxation model. There is no direct interaction of the granular flow with the vibration, and the heap formation is just a steady state which happens to be a stable state to dissipate the input energy.

It is obvious that our model will fail if the dissipation is not strong enough to produce a compact sandpile. In real experiments, when $\Gamma\gg 1$, the heap will disappear and the system becomes gaslike, which means that the dissipation rate of the granular material is not fast enough. In this case, the system is characterized by how the granular flow interacts with the external drive rather than how the energy is dissipated. Such situations occur in the oscillon and wave patterns in experiments of a thin layer under vibration [7] where direct interaction of the granular flow with the vibrating bed is strong. Although our model presented here is for a two-dimensional heap, its heap equation can be trivially extended to three dimensions, and the associated pattern formation problem is very rich.

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APPENDIX

Let $h_0 \equiv h_s(0)$, the first integral of Eq. (11) gives

$$\left(\frac{dh_s}{dx}\right)^2 = \frac{2\beta'}{3}(h_s^3 - h_0^3) - k^2(h_s^2 - h_0), \tag{A1}$$

where the boundary condition $h_s'(0)=0$ is used. A second integration then gives

$$x = \pm\sqrt{\frac{3}{2\beta'}}\int_{h_0}^{h_s}\frac{du}{\sqrt{(u-h_0)[u^2+(h_0-Q)u+(h_0-Q)h_0]}} \tag{A2}$$

where $Q \equiv 3k^2/(2/\beta')$.

Case I: If $Q > h_0$, the above integral can be evaluated in terms of Legendre elliptic integral of the first kind,

$$x = \sqrt{\frac{6}{\beta'(h_0-u_-)}}F(\phi|m), \tag{A3}$$

where $\sin\phi \equiv \sqrt{(h_s-h_0)/(h_s-u_+)}$ and $m \equiv (u_+ - u_-)/(h_0 - u_-)$. F can be inverted in terms of the Jacobian elliptic function sn to give

$$\sin\phi = \mathrm{sn}(\sqrt{\beta'(h_0-u_-)/6}x|m). \tag{A4}$$

Finally $h_s(x)$ can be solved to give

$$h_s(x) = h_0 + (h_0 - u_+)\mathrm{tn}^2(x\sqrt{\beta'(h_0-u_-)/6}|m). \tag{A5}$$

h_0 is to be determined from the boundary condition (3):

$$Na^2 = 2h_0 l + 2(h_0-u_+)\int_0^l \mathrm{tn}^2(\sqrt{\beta'(h_0-u_-)/6}x|m)dx . \tag{A6}$$

The integral can be evaluated using the result

$$\int \frac{dz}{\mathrm{cn}^2(z|m)} = z + \frac{1}{m-1}[E(\mathrm{am}z|m) - \mathrm{dn}(z|m)\mathrm{tn}(z|m)], \tag{A7}$$

where $\mathrm{am}z = \sin^{-1}(\mathrm{sn}z)$ is the amplitude of the Jacobian elliptic function, and E is the elliptic integral of the second kind. Upon integrating, Eq. (A6) becomes

$$Na^2 = 2h_0 l - \frac{12v}{\beta' l}[E(\mathrm{am}v|m) - \mathrm{dn}(v|m)\mathrm{tn}(v|m)], \tag{A8}$$

where $v \equiv l\sqrt{\beta'(h_0-u_-)/6}$. h_0 is solved numerically and checked for the consistency condition $Q > h_0$.

Case II: If $Q < h_0$, then the integral in Eq. (A2) cannot be expressed in terms of a well-studied function, and the improper integral is evaluated numerically. However, in this paper, we consider only $\beta' = \mu k^3$ and thus $Q = 3/(2\mu k)$. Our model is aimed at describing steady heaps that are formed when vibrations are not strong, i.e., small values of k. Therefore, in almost all cases, $Q > h_0$ and the steady state profile is given by Eq. (A5).

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